EXPANSION OF A GAS CLOUD IN VACUUM

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The expansion in vacuum of a gas cloud whose level surfaces are ellipsoids is analyzed. In this case the system of equations of gasdynamics reduces to one of ordinary differential equations of classic mechanics. The exact integral is derived for this system and used for constructing closed solutions of the problem of expansion of a spheroid and of a spinning elliptical cylinder.

In solving the problem of gas cloud expansion considerable use was made of the results of analysis of the motions of an inviscid gas, presented in [1, 3], defined by the affine transformation

$$r_i = F_{ik}a_k \qquad (i,\ k = 1, 2, 3) \tag{1}$$

Here r_i are the coordinates of a gas particle and a_i its Lagrangian coordinates; summation is carried out over recurring subscripts.

Such motions can be considered to be a generalization of the unsteady adiabatic motions of gas investigated earlier by Sedov [2] in which velocity is proportional to the distance from the center of symmetry.

It was first shown in [1] that for motions of the kind defined by (1) the system of equations of gasdynamics reduces to one of nine (in the three-dimensional case) second order ordinary differential equations in elements of matrix F. The same result was later arrived at in [3] in connection with the study of dynamics of a spinning gas cloud. An elegant analysis of the group properties of motions of the kind (1) is given in [3] together with a brief communication of the numerical integration of the system of ordinary differential equations in the particular case of expansion of a gas spheroid initially at rest. A detailed presentation of numerical integration of equations derived in [1] in the case of absence of spinning appears in [4].

The approach to the problem of gas cloud expansion is based here on the system of equations derived in [1, 3] for the elements of matrix F_{ik}. This system is in the form of equations of motion of a particle in a potential field in a nine-dimensional Euclidean space. It will be shown in the following that, when the potential energy is a homogeneous function of a minus two power of coordinates, the equations of mechanics contain in addition to the well-known first integrals a certain supplementary integral. In a problem of gasdynamics such function of potential energy corresponds to a perfect gas without inner degrees of freedom, i. e. of practical interest for the medium considered in this problem. With the use of this additional integral it is possible to obtain an exact solution of the problem of gas cloud expansion in vacuum for a number of initial conditions. In particular, a closed solution defining the expansion of a spheroid in the absence of spinning (numerical solutions exist for this case only) and of a spinning elliptical cylinder is found.

We shall use the same notation as in [3]. We assume the mass of cloud to be unity, and write the system of equations for the matrix elements F_{ik} as

$$F_{ik}^{\cdot\cdot} + \partial U / \partial F_{ik} = 0 \tag{2}$$

where the potential energy $U(F)$ is represented by the internal energy of gas dependent in this case on the determinant of matrix F_{ik}. System (2) has the following seven first integrals [3]

$$\tfrac{1}{2} F_{ik}^{\cdot} F_{ik}^{\cdot} + U = E, \quad F\bar{F}^{\cdot} - F^{\cdot}\bar{F} = J, \quad \bar{F}F^{\cdot} - \bar{F}^{\cdot}F = K \tag{3}$$

where J and K are constant skew-symmetric matrices and $\bar{F}$ is the transpose of matrix F.

Let us now consider a perfect gas without inner degrees of freedom. It is readily seen that in this case the internal energy is a homogeneous function of minus two power of elements of matrix F

$$U = U_0 (\det F_{ik})^{-2/3} \tag{4}$$

For functions U of the form (4) the system (2) has a supplementary integral. To find it we transform expression $F_{ik}F_{ik}$ using (2), Euler's theorem on homogeneous functions and the energy integral (the first of relationships (3)). After simple transformations and integration of the derived equation, we obtain

$$F_{ik}F_{ik} = 2Et^2 + At + B \tag{5}$$

where A and B are constants of integration.

Relationship (5) is the integral of equations of mechanics for an arbitrary potential which is a homogeneous function of minus two power of coordinates. The corresponding central potential $U = \alpha/r^2$ is usually considered in mechanics in connection with the problem of "fall" of a particle onto the center [5]. The conditions of "fall" follow directly from (5), but are usually derived differently. As shown by I. E. Dzialoshinskii, and invariance of the nonstationary Schrödinger equation with potential α/r^2 relative to the group of transformations of the independent variables $r' = \gamma r$ and $t' = \gamma^2 t$ correspond in quantum mechanics to the integral (5). Taking this into consideration it is easy to construct self-similar solutions of the Schrödinger equation with such potential.

Let us apply now integral (5) to solving the problem of gas cloud expansion in vacuum, and first consider the expansion of a nonspinning spheroid. In this case $J = K = 0$ and F_{ik} becomes a diagonal matrix with two independent elements: F_1 and $F_2 = F_3$. As the system of equations for the determination of F_1 and F_2 we can use the energy integral (3) and relationship (5) instead of (2), and write these as

$$\tfrac{1}{2}(F_1^{\cdot 2} + 2F_2^{\cdot 2}) + U_0 (F_1 F_2^2)^{-2/3} = E \tag{6}$$

$$F_1^2 + 2F_2^2 = 2Et^2 + At + B$$

We pass in (6) to polar coordinates σ and ϑ by formulas

$$F_1 = \sigma \sin\vartheta, \qquad F_2\sqrt{2} = \sigma\cos\vartheta$$

and for simplicity shall assume the spheroid to be at rest at $t = 0$. Transformation of (6) yields equality

$$\sigma^2 = 2Et^2 + B$$

and a first order equation for ϑ, the integration of which yields

$$\operatorname{arctg}\left(\frac{2E}{B}\right)^{1/2} t = \int_{\vartheta_0}^{\vartheta} \left[1 - \left(\frac{\sin\vartheta_0 \cos^2\vartheta_0}{\sin\vartheta\cos^2\vartheta}\right)^{2/3}\right]^{-1/2} d\vartheta \tag{7}$$

$$(\vartheta_0 = \vartheta(0))$$

The value $\vartheta_0 = \vartheta_* = \arcsin \tfrac{1}{3}\sqrt{3}$ is critical and corresponds to the expansion of a spherical cloud (the case considered in [2]).

For any arbitrary initial ϑ_0 the variable ϑ in the integral (7) varies between zeros ϑ_0 and ϑ_1 of the radicand. It is not difficult to verify that the limit value of ϑ at $t \to \infty$ lies between ϑ_1 and ϑ_*, which results in the transformation of a cigar-shaped cloud into a disk-shaped one and vice versa as had been noted in [3, 4] on the basis of numerical calculations). For small deviations of the initial shape of a cloud from the spherical we easily obtain the following expression for $\vartheta(t)$:

$$\frac{\vartheta(t)-\vartheta_*}{\vartheta_0-\vartheta_*}=\frac{B-Et^2}{B+Et^2}$$

It is obvious from the last formula that in this case the shape of the cloud at the limit $t\to\infty$ is a conjugate of the initial one and is obtained by substituting $\vartheta_1=2\vartheta_*-\vartheta_0$ for ϑ_0.

For any arbitrary ϑ_0 the integral (7) can be expressed in terms of elliptical integrals of the third kind for which tables exist. The formulas are of the form

$$\operatorname{arctg} t\left(\frac{2E}{B}\right)^{1/2}=\frac{1}{\sqrt{u_0(u_0-u_2)}}\left[\frac{1}{1+u_0}\Pi(\lambda,\mu,p)+\right.$$
$$\left.+2\operatorname{Re}\left(\frac{1+i\sqrt{3}}{1+2u_0-i\sqrt{3}}\Pi(\lambda,\mu_1,p)\right)\right]$$

$$\lambda=\arcsin\left(\frac{u_0-u}{u_0+1}\right)^{1/2},\quad p=\left(\frac{u_0-u_1}{u_0-u_2}\right)^{1/2},\quad \mu=\frac{u_0-u_1}{u_0+1}$$

$$\mu_1=\frac{u_0-u_1}{u_0-{}^1/_2+{}^1/_2 i\sqrt{3}},\qquad u_{1,2}=-\frac{u_0}{2}\pm\left(\frac{u_0^2}{4}-\frac{1}{u_0}\right)^{1/2}$$

$$\Pi(\lambda,\mu,p)=\int_0^{\sin\lambda}\frac{dx}{(1+\mu x^2)\sqrt{(1-x^2)(1-px^2)}},\qquad u_0=\operatorname{tg}^{1/2}\vartheta_0,\quad u=\operatorname{tg}^{1/2}\vartheta$$

Let us now consider the two-dimensional problem of expansion of a spinning infinite elliptical cylinder. The potential energy of a "two-dimensional" perfect gas without inner degrees of freedom is expressed in accepted notation as

$$U=U_0(\det F_{ik})^{-1}\qquad (i,k=1,2)$$

As the equations for matrix F_{ik} we take integrals (3) and (5), and write the system of equations in the form

$$^1/_2(F_{11}^{\cdot 2}+F_{12}^{\cdot 2}+F_{21}^{\cdot 2}+F_{22}^{\cdot 2})+U_0(F_{11}F_{22}-F_{21}F_{12})^{-1}=E$$
$$F_{11}F_{21}^{\cdot}+F_{12}F_{22}^{\cdot}-F_{11}^{\cdot}F_{21}-F_{12}^{\cdot}F_{22}=J$$
$$F_{11}F_{12}^{\cdot}+F_{21}F_{22}^{\cdot}-F_{11}^{\cdot}F_{12}-F_{21}^{\cdot}F_{22}=K$$
$$F_{11}^2+F_{12}^2+F_{21}^2+F_{22}^2=2Et^2+A_1t+B_1$$

Constants J and K are related to the initial angular momentum and vorticity. We introduce new variables σ, ϑ, ξ and η defined by relationships

$$F_{11}=\sigma(\cos\vartheta\cos\xi+\sin\vartheta\cos\eta),\qquad F_{21}=\sigma(\cos\vartheta\sin\xi+\sin\vartheta\sin\eta)$$
$$F_{12}=\sigma(\cos\vartheta\sin\xi-\sin\vartheta\sin\eta),\qquad F_{22}=\sigma(\cos\vartheta\cos\xi-\sin\vartheta\cos\eta)$$

Performing transformation we obtain

$$\sigma^2=Et^2+2At+B,\quad \sigma^2\xi^{\cdot}\cos^2\vartheta=C,\quad \sigma^2\eta^{\cdot}\sin^2\vartheta=D$$

$$\beta\,[\operatorname{arc\,tg}\beta(Et-A)-\operatorname{arc\,tg}\beta A]=\int_{\vartheta_0}^{\vartheta}\left[\beta^2-\frac{C^2}{\cos^2\vartheta}-\frac{D^2}{\sin^2\vartheta}-\frac{U_0}{\cos 2\vartheta}\right]^{-1/2}d\vartheta \quad (8)$$

$$\beta^2=BE-A^2,\quad C={}^1/_4(J-K),\quad D={}^1/_4(J+K)$$

The integral in (8) may be, again, expressed in terms of elliptical integrals; however the related formulas are not adduced here owing to their unwieldiness.

As a very simple particular case of the last problem we shall consider the expansion of a circular cylinder which at the initial instant rotates as a solid at angular velocity ω. Calculations by formula (8) yield

$$F_{11} = F_{22} = (1 + Et^2)^{1/2} \cos [E^{-1/2}\ \omega \operatorname{arctg} (E^{1/2}t)]$$
$$F_{21} = F_{12} = (1 + Et^2)^{1/2} \sin [E^{-1/2}\omega \operatorname{arctg} (E^{1/2}t)]$$

It is not difficult to compute the variation of the (distribution) density of the cloud moment of momentum during expansion. For a Gaussian initial distribution of the gas density $\rho(r, 0) = (2\pi)^{-1} \exp (-{}^1/_2 r^2)e$ we have

$$(xv_y - yv_x)\,\rho(r, t)\,2\pi r\,dr = \omega \exp\left(-\frac{1}{2}\frac{r^2}{1+Et^2}\right)\frac{r^3\,dr}{(1+Et^2)^2}$$

The total momentum $J_{xy} = 2\omega$ obviously remains unchanged. The radius of the layer carrying the maximum moment of momentum varies in time according to the law

$$r_m = [3(1 + Et^2)]^{1/2}$$

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CERTAIN SIMILARITY RELATIONSHIPS FOR THE MOTION OF A GRANULAR COMPACTING MEDIUM

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The solution of one-dimensional self-similar problems of shock wave convergence and of expansion of gas-filled cavities is presented. Conditions of unlimited buildup is derived, two kinds of cavity expansion modes are shown to exist, and the similarity relationship of auger-hole blasting in a uniformly-compacting granular medium and in a fissured rock formation is established.

A class of one-dimensional self-similar problems exists for strong shock waves in gas: concentrated explosion, buildup, short-duration impact, and other (solutions and extensive